

The Decomposition of a Bigraded Left Regular Representation of the Diagonal Action of S_n

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Let $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_{k+1}) = (k, 1^{n-k})$ be a partition of n . In [GH] Garsia and Haiman show that the diagonal action of S_n on the space of harmonic polynomials \mathcal{H}_μ affords the left regular representation ρ of S_n . Furthermore, Garsia and Haiman define a bigraded character of the diagonal action of S_n on \mathcal{H}_μ and show that the character multiplicities are polynomials $\tilde{K}_{\lambda, \mu}(q, t)$ that are closely related to the Macdonald-Kostka polynomials $K_{\lambda, \mu}(q, t)$. In this paper we construct a collection of polynomials $B(\mu)$ that form a basis for \mathcal{H}_μ which exhibits the decomposition of \mathcal{H}_μ into its irreducible parts. Through this connection we give a combinatorial interpretation of the polynomials $\tilde{K}_{\lambda, \mu}(q, t)$. © 1995 Academic Press, Inc.

1. INTRODUCTION

In this writing we will use the French notation for depicting Ferrers diagrams and tableaux. In particular, a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0\}$ gives the successive lengths of the rows of the corresponding Ferrers diagram ordered from bottom to top. We recall that a tableau of shape λ is *injective* if it is a filling of a Ferrers diagram of shape λ with the distinct labels $1, 2, \dots, n$. An injective tableau is *standard* if its entries increase strictly from left to right (west to east) and from bottom to top (south to north). A tableau is said to be *column-strict* if it is a filling of a Ferrers diagram with entries from a set A such that the entries increase weakly from west to east and increase strictly from south to north. On the other hand, a tableau is said to be *row-strict* if it is a filling of a Ferrers diagram with entries from a set A such that the entries increase weakly from south to north and strictly increase from west to east.

We shall assume throughout this writing that μ is a partition of n that is a hook shape, i.e., $\mu = (m, 1^{n-m})$. Let $r_0 = 0$ and let $r_i = \sum_{k=1}^i \mu_k$. Now let

A be the matrix $(a_{i,j}) = (x_i^{\alpha_j} y_i^{\beta_j})$ where α_j is the largest integer k such that $r_k < j$ and $\beta_j = j - r_{\alpha_j} - 1$. For example, if $\mu = (3, 1, 1, 1, 1)$ then

$$A = \begin{pmatrix} x_1^0 y_1^0 & x_1^0 y_1^1 & x_1^0 y_1^2 & x_1^1 y_1^0 & x_1^2 y_1^0 & x_1^3 y_1^0 & x_1^4 y_1^0 \\ x_2^0 y_2^0 & x_2^0 y_2^1 & x_2^0 y_2^2 & x_2^1 y_2^0 & x_2^2 y_2^0 & x_2^3 y_2^0 & x_2^4 y_2^0 \\ x_3^0 y_3^0 & x_3^0 y_3^1 & x_3^0 y_3^2 & x_3^1 y_3^0 & x_3^2 y_3^0 & x_3^3 y_3^0 & x_3^4 y_3^0 \\ x_4^0 y_4^0 & x_4^0 y_4^1 & x_4^0 y_4^2 & x_4^1 y_4^0 & x_4^2 y_4^0 & x_4^3 y_4^0 & x_4^4 y_4^0 \\ x_5^0 y_5^0 & x_5^0 y_5^1 & x_5^0 y_5^2 & x_5^1 y_5^0 & x_5^2 y_5^0 & x_5^3 y_5^0 & x_5^4 y_5^0 \\ x_6^0 y_6^0 & x_6^0 y_6^1 & x_6^0 y_6^2 & x_6^1 y_6^0 & x_6^2 y_6^0 & x_6^3 y_6^0 & x_6^4 y_6^0 \\ x_7^0 y_7^0 & x_7^0 y_7^1 & x_7^0 y_7^2 & x_7^1 y_7^0 & x_7^2 y_7^0 & x_7^3 y_7^0 & x_7^4 y_7^0 \end{pmatrix}.$$

We define Δ_μ to be the determinant of A . It should be noted that the pairs (α_j, β_j) are the coordinates of the cells of the Ferrers diagram of shape μ . For example, with $\mu = (3, 1, 1, 1, 1)$ the Ferrers diagram with the coordinate of each cell as an entry is:

3,0	
2,0	
1,0	
0,0	0,1 0,2

If $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ are sequences of non-negative integers of length n we define

$$\delta_x^p = \delta_{x_1}^{p_1} \delta_{x_2}^{p_2} \dots \delta_{x_n}^{p_n} \quad \text{and} \quad \delta_y^q = \delta_{y_1}^{q_1} \delta_{y_2}^{q_2} \dots \delta_{y_n}^{q_n} \quad (1.1)$$

where δ_{x_i} and δ_{y_j} are the partial derivative operators with respect to x_i and y_j respectively. Now let \mathcal{H}_μ be the linear span of all the partial derivatives of Δ_μ , i.e.

$$\mathcal{H}_\mu = \mathcal{L}[\delta_x^p \delta_y^q \Delta_\mu]_{p,q} \quad (1.2)$$

where p and q run over all nonnegative sequences of length n . Garsia and Haiman in [GH] have shown that for μ a hook shape the dimension of \mathcal{H}_μ as a vector space is equal to $n!$.

Let $\mathbf{R}(X, Y) = \mathcal{Q}[x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n]$. Given $\sigma \in S_n$ we define the diagonal action of σ on the polynomial $P(X, Y) \in \mathbf{R}(X, Y)$ to be

$$\sigma P(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) = P(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n}, y_{\sigma_1}, y_{\sigma_2}, \dots, y_{\sigma_n}). \quad (1.3)$$

Let $\mathcal{H}_{\mu; h, k}$ denote the bihomogeneous component of \mathcal{H}_μ of degree h in x_1, \dots, x_n and degree k in y_1, \dots, y_n . Clearly $\mathcal{H}_{\mu; h, k}$ is invariant under the diagonal action of S_n and thus we will let $\text{char}(\mathcal{H}_{\mu; h, k})$ represent the

character of this S_n action on $\mathcal{H}_{\mu; h, k}$. We define the bigraded character of the diagonal action of S_n on \mathcal{H}_{μ} as

$$\text{char}_{q, t}(\mathcal{H}_{\mu}) = \sum_{h, k \geq 0} t^h q^k \text{char}(\mathcal{H}_{\mu; h, k}). \quad (1.4)$$

Garsia and Haiman in [GH] showed that the diagonal action of S_n on \mathcal{H}_{μ} is equivalent to the regular representation of S_n and that

$$\text{char}_{q, t}(\mathcal{H}_{\mu}) = \sum_{\lambda} \chi^{\lambda} \tilde{K}_{\lambda, \mu}(q, t) \quad (1.5)$$

where $\tilde{K}_{\lambda, \mu}(q, t)$ are polynomials in q and t that are closely related to the Macdonald-Kostka coefficients $K_{\lambda, \mu}(q, t)$ (see [M]). Specifically, we have

$$\tilde{K}_{\lambda, \mu}(q, t) = K_{\lambda, \mu}\left(q, \frac{1}{t}\right) t^{n(\mu)} \quad (1.6)$$

where

$$n(\mu) = \sum_{i=1}^k \mu_i(i-1).$$

Now let $B(\mu)$ be a basis for \mathcal{H}_{μ} such that

$$B(\mu) = \bigcup_{\lambda \vdash n} B_{\lambda}(\mu) \quad (1.7)$$

where $B_{\lambda}(\mu)$ is a collection of bihomogeneous polynomials that span the irreducible representations of shape λ . It is clear from equations 1.4 and 1.5 that

$$\tilde{K}_{\lambda, \mu}(q, t) = f_{\lambda} \sum_{b \in B_{\lambda}(\mu)} t^{|x(b)|} q^{|y(b)|} \quad (1.8)$$

where f_{λ} is the number of standard tableaux of shape λ and $|x(b)|$ and $|y(b)|$ are the degrees of b in the variables x_1, \dots, x_n and y_1, \dots, y_n respectively. Thus an appropriate construction of the set $B_{\lambda}(\mu)$ would lead to a combinatorial interpretation of the coefficients $\tilde{K}_{\lambda, \mu}(q, t)$. It is the construction of such a collection $B(\mu)$ that is the focus of this writing. In particular, we will construct $B_{\lambda}(\mu)$ as a collection of polynomials indexed by a pair (S, C) of tableaux of shape λ where S is standard and C is a *biletter cocharge tableau*. In section 2 we will describe the construction of the biletter cocharge tableaux. In Sections 3 and 4 we will construct the basis $\beta(\mu)$ and prove that it is a basis for \mathcal{H}_{μ} . In Section 5 we will define the diagonal action of S_n on $\beta(\mu)$ and give a combinatorial interpretation of $\tilde{K}_{\lambda, \mu}(q, t)$.

We should also mention that Stembridge [S] has recently also obtained a combinatorial interpretation for the latter coefficients $\tilde{K}_{\lambda, \mu}(q, t)$. His methods are not related to ours and not concerned with the Garsia-Haiman modules.

2. CONSTRUCTION OF THE BILETTER COCHARGE TABLEAUX

Let $A_\mu = \{(0, 0), (0, 1), (0, 2), \dots, (0, m-1), (1, 0), \dots, (n-m-1, 0)\}$ (recall that $\mu = (m, 1^{n-m})$). We will say that $(a_1, a_2) <_{A_\mu} (b_1, b_2)$ if and only if $a_1 - a_2 < b_1 - b_2$. Note that this is a total order on A_μ . To construct the biletter cocharge tableau $C = C(S)$ from a standard tableau S we do the following:

ALGORITHM 2.1.

1. Let $c_m = (0, 0)$.
2. For $i = m-1, \dots, 1$ suppose that $c_{i+1} = (0, s)$. If $i+1$ is north of i in S we then set $c_i = (0, s+1)$. Else let $c_i = c_{i+1} = (0, s)$.
3. For $i = m+1, \dots, n$ suppose that $c_{i-1} = (s, 0)$. If i is north of $i-1$ in S we then set $c_i = (s+1, 0)$. Else let $c_i = c_{i-1} = (s, 0)$.
4. Now let $C = C(S)$ be the tableau that has entry c_i in the same cell that had entry i in S .

EXAMPLE. Suppose that $\mu = 5, 1^4$. and that

$$S = \begin{array}{ccc} 7 & 9 & \\ 3 & 6 & \\ 2 & 5 & \\ 1 & 4 & 8 \end{array}. \quad (2.1)$$

Now $c_5 = (0, 0)$. Since 5 is north of 4 in S we set $c_4 = (0, 1)$. 3 is north of 4 so that $c_3 = (0, 1)$. In the same manner $c_2 = (0, 2)$ and $c_1 = (0, 3)$. Continuing with step 3, 6 is north of 5 and thus $c_6 = (1, 0)$, etc. Thus the biletter cocharge tableau $C = C(S)$ is

$$C = \begin{array}{ccc} (2, 0) & (3, 0) & \\ (0, 1) & (1, 0) & \\ (0, 2) & (0, 0) & \\ (0, 3) & (0, 1) & (2, 0) \end{array} \quad (2.2)$$

Analogous to the biletter cocharge tableaux $C = C(S)$ are the tableaux $K = K(S)$ and $L = L(S)$ that are constructed in the following manner. First we let $\{a_1, a_2, \dots, a_n\}$ be the elements of the collection A_μ listed in increasing order. We define $K = K(S)$ to be the tableau that has entry a_i in the same cell that had entry i in S . We define $L = L(S)$ to be the tableau that has entry $a_i - c_i$ in the cell of S that contains i where c_i is as given in algorithm 2.1.

EXAMPLE. With S given in 2.1, we have $a_1 = (0, 4)$, $a_2 = (0, 3)$, etc. Thus

$$K = K(S) = \begin{array}{ccc} (2, 0) & (4, 0) & \\ (0, 2) & (1, 0) & \\ (0, 3) & (0, 0) & \\ (0, 4) & (0, 1) & (3, 0) \end{array} \quad (2.3)$$

and

$$L = L(S) = \begin{pmatrix} (0, 0) & (1, 0) \\ (0, 1) & (0, 0) \\ (0, 1) & (0, 0) \\ (0, 1) & (0, 0) & (1, 0) \end{pmatrix} \quad (2.4)$$

It should be noted that both $C = C(S)$ and $K = K(S)$ are column-strict and that $L = L(S)$ and $K = K(S)$ are row-strict based on the ordering $<_{A_\mu}$. Given $C = C(S)$, $K = K(S)$ or $L = L(S)$ it is not difficult to reconstruct the standard tableau S from which it was constructed and hence the collections $\{C(S)\}_S$, $\{K(S)\}_S$ and $\{L(S)\}_S$ are in a one-to-one correspondence with the standard tableaux.

3. CONSTRUCTION OF THE COLLECTIONS $B(\mu)$ AND $B_\lambda(\mu)$

Given a column-strict tableau T we say that the *row sequence* of T (denoted by $rs(T)$) is the word obtained by reading the entries of T from left to right, row by row, starting at the bottom. The type $\tau(T)$ is the rearrangement of the letters of $rs(T)$ in decreasing order with respect to $<_{A_\mu}$. Now let $>_L$ represent the lexicographic order. If S and T are two column-strict tableau we will say that $S >_{tr} T$

- (a) if $\tau(S) >_L \tau(T)$; or
- (b) if $\tau(S) = \tau(T)$ and $rs(S) >_L rs(T)$.

Suppose that S and T are standard tableaux of shape λ and that $C = C(T)$, $K = K(T)$ and $L = L(T)$. Let $d_i = (d_{i,1}, d_{i,2}) \in A_\mu$ and $c_i = (c_{i,1}, c_{i,2}) \in A_\mu$ be the entry in the corresponding cells of L and C respectively that contain i in S . Let us set

$$\begin{aligned} M(S, C) &= \prod_{1 \leq i \leq n} (i \mid c_i) \\ D(S, L) &= \prod_{1 \leq i \leq n} (i \mid d_i) \end{aligned} \quad (3.1)$$

where $(i \mid c_i) = x_i^{c_{i,1}} y_i^{c_{i,2}}$ and $(i \mid d_i) = x_i^{d_{i,1}} y_i^{d_{i,2}}$. For example, suppose that

$$S = \begin{pmatrix} 8 & 9 \\ 6 & 7 \\ 4 & 5 \\ 1 & 2 & 3 \end{pmatrix}$$

and that C and L are given in 2.2 and 2.4, i.e.

$$C = \begin{pmatrix} (2, 0) & (3, 0) \\ (0, 1) & (1, 0) \\ (0, 2) & (0, 0) \\ (0, 3) & (0, 1) & (2, 0) \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} (0, 0) & (1, 0) \\ (0, 1) & (0, 0) \\ (0, 1) & (0, 0) \\ (0, 1) & (0, 0) & (1, 0) \end{pmatrix}$$

Then

$$M(S, C) = (1 \mid (0, 3)) (2 \mid (0, 1)) (3 \mid (2, 0)) (4 \mid (0, 2)) (5 \mid (0, 0)) \\ \times (6 \mid (0, 1)) (7 \mid (1, 0)) (8 \mid (2, 0)) (9 \mid (3, 0))$$

and

$$D(S, L) = (1 \mid (0, 1)) (2 \mid (0, 0)) (3 \mid (1, 0)) (4 \mid (0, 1)) (5 \mid (0, 0)) \\ \times (6 \mid (0, 1)) (7 \mid (0, 0)) (8 \mid (0, 0)) (9 \mid (1, 0))$$

Let we will define $\delta_i^{(a, b)} = \delta_{x_i}^a \delta_{y_i}^b$. For $\sigma \in S_n$ we define

$$\sigma(\delta_1^{a_1} \delta_2^{a_2} \cdots \delta_n^{a_n}) = \delta_{\sigma(1)}^{a_1} \delta_{\sigma(2)}^{a_2} \cdots \delta_{\sigma(n)}^{a_n}.$$

Let

$$H = H(X, Y)$$

be a polynomial and suppose that

$$H = \sum_h c_h h(X, Y)$$

where each $h(X, Y)$ is a monomial. We define

$$\delta^H = \sum_h c_h \delta^h.$$

Note that if $G = G(X, Y) \in \mathbf{R}(X, Y)$ then

$$\sigma(\delta^H G) = \sigma(\delta^H) \sigma(G).$$

Let E_1, E_2, \dots, E_k be the collections of entries in columns 1, 2, ..., k of S respectively and let S_{E_i} be the symmetric group acting on the collection E_i . We define

$$S_E = S_{E_1} \times S_{E_2} \times \cdots \times S_{E_k} \\ (S \mid C) = \sum_{\sigma \in S_E} \text{sgn}(\sigma) \sigma M(S, C) \quad (3.2) \\ P(S, L) = \sum_{\sigma \in S_E} \sigma D(S, L).$$

It should be noted that the polynomial $(S | C)$ is the *bideterminant* of (S, C) (see [DKR]). The following theorem is the crucial result that links the polynomials $(S | C)$ and $P(S, T)$.

THEOREM 3.1. *Suppose S and T are standard tableaux of the same shape and that $L = L(T)$ and $C = C(T)$. Then*

$$\delta^{P(S, L)} A_\mu = c(S | C) + \sum_{C' >_u C} c_{S, C'}(S | C') \quad (3.3)$$

where $c \neq 0$.

Before we prove theorem 3.1 we will need additional definitions and results. As before let c_i, k_i and d_i be the entries in $C = C(T)$, $K = K(T)$ and $L = L(T)$ that are in the same cell as i in S . Let now $D = D(S, L)$ and

$$Q = Q(S, K) = (1 | k_1)(2 | k_2) \cdots (n | k_n).$$

If $\gamma, \phi \in S_n$ then we clearly have

$$\begin{aligned} \gamma D &= (\gamma(1) | d_1)(\gamma(2) | d_2) \cdots (\gamma(n) | d_n) \\ &= (1 | d_{\gamma^{-1}(1)})(2 | d_{\gamma^{-1}(2)}) \cdots (n | d_{\gamma^{-1}(n)}) \\ \phi Q &= (\phi(1) | k_1)(\phi(2) | k_2) \cdots (\phi(n) | k_n) \\ &= (1 | k_{\phi^{-1}(1)})(2 | k_{\phi^{-1}(2)}) \cdots (n | k_{\phi^{-1}(n)}) \end{aligned} \quad (3.4)$$

Define $W(S, \phi K - \gamma L)$ to be the tableau that has entry $k_{\phi(i)} - d_{\gamma(i)} \in A_\mu$ in the same cell that has entry i in S . We now have the following facts:

LEMMA 3.2. *Let S and T be standard tableaux and let $C = C(T)$, $K = K(T)$ and $L = L(T)$. Suppose that $\phi \in S_n$, $D = D(S, L)$, $Q = Q(S, K)$, $W_1 = W(S, \phi^{-1}K - L)$ and $W_2 = W(S, \phi^{-1}K - \phi^{-1}L)$. Then*

(a)

$$\sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\delta^D(\phi Q)) = c_{S, L}(S | W_1)$$

with the convention that if any of the coefficients of $k_{\phi^{-1}(i)} - d_i$ are negative then $(S | W_1) = 0$.

(b) If $\phi \delta^D = \delta^D$ and $\phi \in S_E$ then

$$\text{sgn}(\phi) \sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\delta^D(\phi Q)) = c_{S, L}(S | C)$$

where $c_{S, L} > 0$.

(c) Suppose $\phi\delta^D = \delta^D$. Then

$$\sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\delta^D(\phi Q)) = c_{S, L}(S \mid W_2)$$

(d) Suppose that $\phi\delta^D = \delta^D$ and $\phi \notin S_E$. Let W'_2 be the tableau that we get by rearranging the entries of each column of $W_2 = W(S, \phi^{-1}K - \phi^{-1}L)$ in increasing order. If $(S \mid W_2) \neq 0$ then W'_2 is a column strict tableau with respect to $<_{A_\mu}$ and $rs(W'_2) > rs(C)$.

(e) If $\phi\delta^D \neq \delta^D$ then $\tau(W_1) > \tau(C)$.

Proof. (a) Suppose that for $1 \leq i \leq n$ all of the terms $k_{\phi^{-1}(i)} - d_i$ are nonnegative. Then

$$\begin{aligned} \delta^D(\phi Q) &= (\delta_1^{d_1} \delta_2^{d_2} \cdots \delta_n^{d_n})(1 \mid k_{\sigma^{-1}(1)})(2 \mid k_{\sigma^{-1}(2)}) \cdots (n \mid k_{\sigma^{-1}(n)}) \\ &= c'_{S, L}(1 \mid k_{\sigma^{-1}(1)} - d_1)(2 \mid k_{\sigma^{-1}(2)} - d_2) \cdots (n \mid k_{\sigma^{-1}(n)} - d_n) \end{aligned} \quad (3.5)$$

and therefore

$$\sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\delta^D \phi Q) = c_{S, L}(S \mid W_1).$$

(b) With $\phi\delta^D = \delta^D$ and $\phi \in S_E$ we have

$$\begin{aligned} \text{sgn}(\phi) \sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\delta^D(\phi Q)) &= \text{sgn}(\phi) \sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\phi(\delta^D)(\phi Q)) \\ &= \text{sgn}(\phi) \sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma\phi(\delta^D Q) \\ &= \text{sgn}(\phi) \text{sgn}(\phi) \sum_{\beta \in S_E} \text{sgn}(\beta) \beta(\delta^D Q) \end{aligned}$$

Note that $\delta^D Q = c_{S, L} M(S, C)$ and thus

$$\sum_{\beta \in S_E} \text{sgn}(\beta) \beta(\delta^D Q) = c_{S, L}(S \mid C)$$

(c) If $\phi\delta^D = \delta^D$ then $\delta^D(\phi Q) = \phi(\delta^D)(\phi Q)$ and thus we have

$$\begin{aligned} (\phi\delta^D)(\phi Q) &= c_{S, L}(1 \mid k_{\phi^{-1}(1)} - d_{\sigma^{-1}(1)}) \\ &\quad \times (2 \mid k_{\phi^{-1}(2)} - d_{\sigma^{-1}(2)}) \cdots (n \mid k_{\sigma^{-1}(n)} - d_{\sigma^{-1}(n)}) \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\delta^D(\phi Q)) &= \sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\phi(\delta^D)(\phi Q)) \\ &= c_{S, L}(S \mid W_2) \end{aligned}$$

(d) Recall that L is row-strict and K is column-strict. Suppose that $(S \mid W_2) \neq 0$. Note that $(S \mid W_2) = \pm (S \mid W'_2)$ where W'_2 is strictly increasing in its columns. Thus to prove that W'_2 is column strict it is enough to show that $k_i - d_i \leq_{A_\mu} k_j - d_j$ whenever $d_i <_{A_\mu} d_j$. If $k_i <_{A_\mu} k_j \leq_{A_\mu} (0, 0)$ then $k_i = k_j + (0, j - i)$ and $d_i \geq_{A_\mu} d_j + (0, j - i)$. Therefore we have that $k_i - d_i \leq_{A_\mu} k_j - d_j$. On the otherhand if $(0, 0) \leq_{A_\mu} k_i <_{A_\mu} k_j$ then $k_i + (j - i, 0) = k_j$ and $d_i + (j - i, 0) \geq_{A_\mu} d_j$. Thus $k_i - d_i \leq_{A_\mu} k_j - d_j$. Finally, if $k_i \leq_{A_\mu} (0, 0) \leq_{A_\mu} k_j$ it is clear that $k_i - d_i \leq_{A_\mu} k_j - d_j$.

Furthermore, there exists a $\gamma \in S_E$ such that $\gamma W_2 = W'_2$ and $\gamma(\delta^D) = \delta^D$. Let w_i and c_i be the entries in the cells of W'_2 and C that contain i in S . Now let c_j and w_j be the first location in the row sequence order such that $c_j \neq w_j$. For some i , $c_j = k_i - d_i$ and $w_j = k_{\gamma(i)} - d_{\gamma(i)} = k_{\gamma(i)} - d_i$. Since $c_m = w_m$ for $1 \leq m \leq j - 1$ we have $k_{\gamma(i)} >_{A_\mu} k_i$. Thus $w_j > c_j$ and $rs(W'_2) > rs(C)$.

(e) Let κ_i and λ_i be the entries in the cells of K and L that contain i in T . (Recall that k_i and d_i are indexed by the cell that contain i in S .) Now $\tau(C) = \kappa_n - \lambda_n, \kappa_{n-1} - \lambda_{n-1}, \dots, \kappa_1 - \lambda_1$. Furthermore, $\kappa_p > \kappa_q$ and $\lambda_p \geq \lambda_q$ for $p > q$. By the definition of $\tau(W_1)$ and 3.2 part (a) we see that $\tau(W_1)$ is the rearrangement of

$$\{\kappa_{\phi^{-1}(1)} - \lambda_1, \kappa_{\phi^{-1}(2)} - \lambda_2, \dots, \kappa_{\phi^{-1}(n)} - \lambda_n\}$$

in decreasing order. But this is the same as the rearrangement of

$$\{\kappa_1 - \lambda_{\phi(1)}, \kappa_2 - \lambda_{\phi(2)}, \dots, \kappa_n - \lambda_{\phi(n)}\}$$

in decreasing order. Now let m be the largest integer such that $\lambda_m \neq \lambda_{\phi(m)}$. Since $\lambda_s = \lambda_{\phi(s)}$ for $s > m$ we have $\lambda_m > \lambda_{\phi(m)}$ and thus $\kappa_m - \lambda_m < \kappa_m - \lambda_{\phi(m)}$. Therefore $\tau(W_1) > \tau(C)$. ■

We may now proceed with the proof of theorem 3.1.

Proof of Theorem 3.1. Recall that

$$A_\mu = \pm \sum_{\sigma \in S_n} \text{sgn}(\sigma) \sigma Q$$

Thus

$$\begin{aligned} \pm \delta^{P(S, T)} A_\mu &= \sum_{\sigma \in S_E} (\sigma \delta^D) \sum_{\phi \in S_n} \text{sgn}(\phi) (\phi Q) \\ &= \sum_{\phi \in S_n} \sum_{\sigma \in S_E} \text{sgn}(\phi) (\sigma \delta^D) (\phi Q) \\ &= \sum_{\beta \in S_n / S_E} \text{sgn}(\beta) \sum_{\gamma \in S_E} \sum_{\sigma \in S_E} \text{sgn}(\gamma) (\sigma \delta^D) (\gamma \beta Q) \\ &= \sum_{\beta \in S_n / S_E} \text{sgn}(\beta) \sum_{\gamma \in S_E} \sum_{\sigma \in S_E} \text{sgn}(\gamma) (\gamma (\gamma^{-1} \sigma \delta^D)) (\gamma (\beta Q)) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\beta \in S_n/S_E} \text{sgn}(\beta) \sum_{\gamma \in S_E} \sum_{\sigma \in S_E} \text{sgn}(\gamma) (\gamma(\sigma\delta^D))(\gamma(\beta Q)) \\
&= \sum_{\beta \in S_n/S_E} \text{sgn}(\beta) \sum_{\sigma \in S_E} \sum_{\gamma\sigma^{-1} \in S_E} \text{sgn}(\sigma^{-1}) \\
&\quad \times \text{sgn}(\gamma) (\gamma\sigma^{-1}(\sigma\delta^D))(\gamma\sigma^{-1}(\beta Q)) \\
&= \sum_{\beta \in S_n/S_E} \text{sgn}(\beta) \sum_{\sigma \in S_E} \text{sgn}(\sigma^{-1}) \sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\delta^D) \gamma(\sigma^{-1}\beta Q) \\
&= \sum_{\phi \in S_n} \text{sgn}(\phi) \sum_{\gamma \in S_E} \text{sgn}(\gamma) \gamma(\delta^D(\phi Q)) \tag{3.6}
\end{aligned}$$

Lemma 3.2(b) implies that $c \neq 0$ in equation 3.3. Lemmas 3.2(c), (d), and (e) imply the rest. ■

For standard tableaux S and T we define

$$II[S, T] = \delta^{P(S, L)} \Delta_\mu \tag{3.7}$$

where $L = L(T)$ and

$$B(\mu) = \{II[S, T]\}_{S, T} \tag{3.8}$$

where S and T run over all pairs of standard tableaux of the same shape. Note that there are $n!$ elements in $B(\mu)$. Thus to prove that the set $B(\mu)$ is a basis for \mathcal{H}_μ all we need to show is that they are linearly independent. This is the subject of the next section.

4. INDEPENDENCE OF THE COLLECTION $B(\mu)$

In order to show that the polynomials that we constructed are linearly independent we need to recall the definition of a *Capelli Operator* (see [DKR]). For a given $a_i \in A$ let us set $J(V, a_i) = \{j_1, j_2, \dots, j_{\alpha_i}\}$ to be the integers such that $z_{j_k} = a_i$ in $V = (1 \mid z_1)(2 \mid z_2) \cdots (n \mid z_n)$. For an integer m we define $\mathcal{J}(V, a_i, m)$ to be the collection of all m -subsets of $J(V, a_i)$. We define the action of the *set polarization operator* $D^m(t_q, a_i)$ on V as follows:

- (a) if $\alpha_i < m$ then $D^m(t_q, a_i) V = 0$
- (b) if $\alpha_i \geq m$ then

$$D^m(t_q, a_i) V = \sum_{C \in \mathcal{J}(V, a_i, m)} \prod_{k \notin C} (k \mid z_k) \prod_{k \in C} (k \mid a_i)$$

Now linearly extend this definition to all polynomials of $\mathbf{R}(X, Y)$. For example, if

$$V = (1 \mid (0, 0)) (2 \mid (1, 0)) (3 \mid (1, 0)) (4 \mid (1, 1)) (5 \mid (1, 0)) \\ \times (6 \mid (1, 2)) (7 \mid (2, 0)) (8 \mid 1, 0))$$

then

$$D^3((1, 0), t) V = (1 \mid (0, 0)) (2 \mid t) (3 \mid t) (4 \mid (1, 1)) (5 \mid t) (6 \mid (1, 2)) (7 \mid (2, 0)) (8 \mid 1, 0)) \\ + (1 \mid (0, 0)) (2 \mid t) (3 \mid t) (4 \mid (1, 1)) (5 \mid (1, 0)) (6 \mid (1, 2)) (7 \mid (2, 0)) (8 \mid t) \\ + (1 \mid (0, 0)) (2 \mid t) (3 \mid (1, 0)) (4 \mid (1, 1)) (5 \mid t) (6 \mid (1, 2)) (7 \mid (2, 0)) (8 \mid t) \\ + (1 \mid (0, 0)) (2 \mid (1, 0)) (3 \mid t) (4 \mid (1, 1)) (5 \mid t) (6 \mid (1, 2)) (7 \mid (2, 0)) (8 \mid t).$$

Let us assume that T is a standard tableau of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0)$. Furthermore, let $\alpha_i(q)$ and $\beta_j(q)$ be the number of occurrences of x_i and a_j in row q of T and T' respectively. Given a bitableau $[T, T']$, the *Capelli operator* $Ca(T, T')$ is defined as

$$Ca(T, T') = \prod_{1 \leq q \leq k} \left[\left(\prod_{1 \leq i \leq n} D^{\alpha_i(q)}(t_q, x_i) \right) \left(\prod_{1 \leq j \leq m} D^{\beta_j(q)}(s_q, a_j) \right) \right]. \quad (4.1)$$

For example, if

$$[T, T'] = \begin{bmatrix} 4 & 7 & (0, 1) & (0, 1) \\ 3 & 5 & (1, 0) & (1, 0) \\ 1 & 2 & 6 & (0, 0) & (0, 0) & (1, 0) \end{bmatrix}$$

then

$$Ca(T, T') = D^1(t_1, 1) D^1(t_1, 2) D^1(t_1, 6) D^1(t_2, 3) D^1(t_2, 5) D^1(t_3, 4) \\ D^1(t_3, 7) D^2(s_1, (0, 0)) D^1(s_1, (1, 0)) D^2(s_2, (1, 0)) D^2(s_3, (0, 1)).$$

In [DKR], theorems 4.1 and 4.2 are proved:

THEOREM 4.1. *Let S and U be standard tableaux and let T and V be tableaux. If the $\tau(T) \neq \tau(V)$ then $Ca(S, T) (U \mid V) = 0$.*

THEOREM 4.2. *Let S and U be standard tableaux and let T and V be column-strict tableaux. Furthermore, let us assume that S and T have shape λ and that U and V have shape ν . Let $(S \mid T)$ and $(U \mid V)$ be two bideterminants of the same type. Then*

- (a) $Ca(S, T)(S \mid T) \neq 0$;
- (b) if v is higher then $Ca(S, T)(U \mid V) = 0$;
- (c) If $v = \lambda$ and if $Ca(S, T)(U \mid V) \neq 0$ then $[U, V]$ is smaller than $[S, T]$ in its associated column sequences.

Now, let us suppose that U and V are column-strict tableaux and let $\lambda(U)$ and $\lambda(V)$ be the shapes of U and V respectively. We will say that $U >_{str} V$ if and only if

- (a) $\lambda(U) >_L \lambda(V)$; or
- (b) $\lambda(U) = \lambda(V)$ and $U >_{lr} V$.

We can now prove the following:

THEOREM 4.3. *The collection of polynomials in $B(\mu)$ is linearly independent and hence they form a basis for \mathcal{H}_μ .*

Proof. Suppose that

$$\sum_{II[S, T] \in B(\mu)} c_{S, T} II[S, T] = 0. \quad (4.2)$$

Recall that theorem 3.1 implies that

$$II[S, T] = \delta^{P(S, T)} A_\mu = c_{S, C}(S \mid C) + \sum_{W >_r C} c_w(S \mid W)$$

where $C = C(T)$ and $c_{S, C} \neq 0$. Let $[S', T']$ be the pair of tableaux in equation 4.2 that is the smallest with respect to $>_{str}$ such that $c_{S', T'} \neq 0$. Let $C' = C(T')$. Now

$$\begin{aligned} & Ca(S', C') II[S', T'] \\ &= c_{S', C'} Ca(S', C')(S' \mid C') + \sum_{W >_r C'} c_w Ca(S', C')(S' \mid W) \\ &= c_{S', C'} Ca(S, C')(S' \mid C') \\ &\neq 0 \end{aligned} \quad (4.3)$$

by theorems 3.1, 4.1 and 4.2. Theorems 3.1, 4.1 and 4.2 also imply that for $[S, T] >_{str} [S', T']$

$$Ca(S', C') II[S, T] = 0.$$

Hence if we apply the Capelli operator $Ca(S', C')$ to equation 4.2, the equation reduces to

$$c_{S', T'} c_{S', C'} Ca(S', C') II[S', T'] = 0.$$

Equation 4.3 implies that $c_{S', C'} Ca(S', C') \Pi[S', T'] \neq 0$, but this implies that $c_{S', T'} = 0$ a contradiction. Thus the collection $B(\mu)$ is linearly independent. ■

5. THE DIAGONAL ACTION OF S_n ON $B(\mu)$

Let now $T_1, T_2, \dots, T_{n_\lambda}$ be the standard tableaux of shape λ lexicographically ordered with respect to their row sequence. Let $R_1(T_i) R_2(T_i) \dots R_k(T_i)$ and $C_1(T_i) C_2(T_i) \dots C_m(T_i)$ be the row decomposition and the column decomposition of T_i respectively. Let us define

$$S_{R(T_i)} = S_{R_1(T_i)} \times \dots \times S_{R_k(T_i)}$$

and

$$S_{C(T_i)} = S_{C_1(T_i)} \times \dots \times S_{C_m(T_i)}.$$

We set

$$P(T) = \sum_{\sigma \in S_{R(T_i)}} \sigma$$

and

$$N(T) = \sum_{\sigma \in S_{C(T_i)}} \text{sign}(\sigma) \sigma.$$

For the standard tableau T_j let $C = C(T_j)$ and $L = L(T_j)$. Let us define

$$E(T_i, C) = \sum_{\phi \in S_n} \text{sgn}(\phi) \delta^{D(T_i, L)}(\phi Q)$$

Theorem 4.3 implies that $E(T_i, C) \neq 0$. Now,

$$\begin{aligned} \delta^{D(T_i, L)} A_\mu &= \sum_{\gamma \in S_T} \text{sgn}(\gamma) \gamma \left(\sum_{\phi \in S_n} \text{sgn}(\phi) \delta^{D(T_i, L)}(\phi Q) \right) \\ &= N(T_j) E(T_j, C). \end{aligned}$$

For any injective tableau T of shape λ a use of the Rota straightening algorithm (see [DKR]) yields

$$N(T) = \sum_i a_{T_j, T_i}(\sigma) N(T_i) \sigma_{T_i, T} + \sum_j b_{T_j, V_j} N(V_j) \quad (5.1)$$

where each V_j is a standard tableau such that $\lambda(V_j) >_L \lambda(T)$. Recall that if $\lambda(V_k) >_L \lambda(T)$ then $N(V_k)P(T_j) = 0$. Thus multiplying equation 5.1 on the right by $P(T)\sigma_{T,T_j}$ yields

$$\begin{aligned} N(T)P(T)\sigma_{T,T_j} &= \sum_i a_{T,T_i}(\sigma) N(T_i)\sigma_{T_i,T}P(T)\sigma_{T,T_j} \\ &= \sum_i a_{T,T_i}(\sigma) N(T_i)\sigma_{T_i,T_j}P(T_j). \end{aligned}$$

The matrices $A^\lambda(\sigma) = (a_{T,T_i}(\sigma))$ can be shown in arguments similar to those found in [A] and [GW] to define the S_n irreducible representation of shape λ . But, from 5.1 we have

$$\begin{aligned} \sigma_{T,T_i}N(T_i)E(T_i, C) &= N(T)E(T, C) \\ &= \sum_i a_{T,T_i}(\sigma) N(T_i)\sigma_{T_i,T}E(T, C) + \sum_j b_{T_j,V_j}N(V_j)E(T, C) \\ &= \sum_i a_{T,T_i}(\sigma) N(T_i)\sigma_{T_i,T}E(T, C) + \sum_j b_{T_j,V_j}N(V_j)E(T, C) \\ &= \sum_i a_{T,T_i}(\sigma) N(T_i)E(T_i, C) + \sum_j b_{T_j,V_j}N(V_j)E(T, C). \end{aligned} \quad (5.2)$$

Summarizing these results we have the following theorem

THEOREM 5.1. *Let T_i and T_j be standard tableaux of shape λ . Then for any $\sigma \in S_n$ we have*

$$\sigma\Pi[T_i, T_j] = \sum_{T_k} a_{T_k, T_j} \Pi[T_k, T_j] + \sum_{C(V) >_{str} C(T_j)} b_{U, V} \Pi[U, V]$$

where T_k ranges over all standard tableaux of shape λ , the coefficients a_{T_k, T_j} only depend on T_k and T_j and the matrices $A^\lambda = (a_{T_k, T_j})$ define Young's natural representation of shape λ . Thus the matrices of the diagonal action of S_n on the collection $B(\mu)$ ordered by $>_{str}$ form an upper block triangular version of the regular representation.

Thus we have shown the decomposition of $B(\mu)$ needed for 1.7 and we have proven 1.8. We also have the following corollary:

COROLLARY 5.2.

$$\tilde{K}_{\lambda, \mu}(q, t) = f_\lambda \sum_C q^{x(C)} t^{y(C)}$$

where f_λ is the number of standard tableaux of shape λ and the summation is over all billetter cocharge tableaux C of shape λ and where $x(C)$ and $y(C)$ are the sums of the first and second coordinates of the entries in C , respectively.

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